Hirota method for the nonlinear Schrödinger equation with an arbitrary linear time-dependent potential

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In this paper, a Hirota method is developed for applying to the nonlinear Schrödinger equation with arbitrary time-dependent linear potential which denotes the dynamics of soliton solutions in quasi-one-dimensional Bose-Einstein condensation. The nonlinear Schrödinger equation is decoupled to two equations carefully. With a reasonable assumption the one- and two-soliton solutions are constructed analytically in the presence of an arbitrary time-dependent linear potential.

I.   INTRODUCTION

The realization of Bose-Einstein condensation (BEC) [1, 2] which strongly stimulates the exploration of nonlinear properties of matter waves have opened a new field of nonlinear atom optics, such as four wave mixing in BEC's [3], the study of various types of excitations. One of particular interest is macroscopically excited Bose-Einstein condensed states, such as vortices [4–6] and solitons [7–13]. The existence of solitonic solutions is a general feature of nonlinear wave equations. For the case of an atomic Bose-Einstein condensate, the macroscopic wave function of the condensate obeys the so-called Gross-Pitaevskii (G-P) equation, whose nonlinearity result from the interatomic interactions. It is well known that the G-P equation has of either dark or bright solitons depending on the repulsive or attractive nature of the interatomic interactions, respectively. A dark soliton [14, 15] in BEC is a macroscopic excitation of the condensate which is characterized by a local density minimum and a phase gradient of the wave function at the position of the minimum. A

bright soliton [16–18] in BEC is expected for the balance between the dispersion and the attractive mean-field energy. Several methods have been applied to obtain the soliton solutions of G-P equation with different potential [9–13, 19, 20], as well as the dynamics of the excitation of the condensate was discussed. When the longitudinal dimension of the BEC is much longer than its transverse dimensions which is the order of its healing length, the G-P equation can be reduced to the quasi-one-dimensional (quasi-1D) regime. This trapped quasi-low-dimensional [21] condensates has offered an useful tool for investigating the nonlinear excitations such as solitons and vortices, which are more stable than in 3D, where the solitons suffer from the transverse instability and the vortices can bend. Thus the study of both theory and experiment is very important for the soliton excitations in quasi-low-dimensional BECs.

In this paper, we consider the mean-field model of a quasi-1D BEC trapped in a linear time-dependent potential which is given by

$$i\hbar \frac{\partial}{\partial T}\Psi = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\Psi + Xf(T)\Psi + g\left|\Psi\right|^2\Psi, \tag{1}$$

where $\int \left|\Psi\right|^2 dX = N$ is the number of atoms in the condensate, $f(T)$ is the arbitrary function of time T, the interacting constant of two-atom is given by $g = 2\hbar^2 a/ml_0^2$ ([22]) with m the mass of the atom, a the s-wave scattering length ($a > 0$ for repulsive interaction; while $a < 0$ for attractive interaction), and $l_0 \equiv \sqrt{\hbar/m\omega_0}$ denoting the characteristic length extension of the ground state wave function of harmonic oscillator. Making a dimensionless transformation, we can rewrite equation (1) as

$$i\frac{\partial}{\partial t}\psi + \frac{1}{2}\frac{\partial^2}{\partial x^2}\psi + xf(t)\psi + \mu\left|\psi\right|^2\psi = 0, \tag{2}$$

where x is measured in units of l_0, t in units of $ml_0^2/\hbar$, ψ in units of the square root of Nl_0, the interaction constant μ is defined as $\mu = -2Nl_0a$, and $f(t) = -\frac{ml_0^3}{\hbar^2}f(\frac{t}{\hbar/ml_0^2})$. The exact soliton solutions of Eq. (2) can been constructed by the inverse scattering method [23] and F-expansion method [24]. It should be noted that in the absence of the linear potential, i.e. $xf(t) = 0$, a Hirota method can be applied to Eq. (2) directly for getting the bright and dark soliton solutions. However, with the consideration of the linear potential the Hirota method should be developed carefully. This is our purpose in the present paper. With a reasonable assumption we demonstrate how to construct the exact one- and two-soliton solutions of Eq. (2) in terms of this developed technique.

II.   ONE-SOLITON SOLUTION

Now we introduce the main idea of the Hirota method briefly. Firstly, it apply a direct transformation to the nonlinear equation. Then in terms of the reasonable assumption the nonlinear equation can be decoupled to two equations from which the one- and two-soliton solutions can be constructed effectively. To this purpose we consider the following transformation

$$\psi = \frac{G(x,t)}{F(x,t)}, \tag{3}$$

where $G(x,t)$ is complex function and $F(x,t)$ is a real function. With this transformation Eq. (2) becomes

$$F\left(iD_t + \frac{1}{2}D_x^2\right)G \cdot F + G \cdot F^2 \cdot xf(t) - G(\frac{1}{2}D_x^2 F \cdot F - \mu\overline{G}G) = 0, \tag{4}$$

where the overbar denotes the complex conjugate, D_t and D_x^2 are called the Hirota bilinear operators defined as

$$D_x^m D_t^n G(x,t) \cdot F(x,t) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n G(x,t) F(x',t')\,|_{x=x',t=t'}. \tag{5}$$

If the term $xf(t) = 0$ Eq. (4) reduces to the normal nonlinear equation which can be decoupled easily to two equations. In the presence of the term $xf(t)$ we should deal with Eq. (4) carefully. Many attempts show that Equation (4) can be decoupled as

$$\frac{1}{2}D_x^2 F \cdot F - \mu\overline{G}G = 0,$$
$$\left(iD_t + \frac{1}{2}D_x^2 + xf(t)\right)G \cdot F = 0, \tag{6}$$

in which the spatial and time dependence term $xf(t)$ will give a difficulty for getting solutions as shown below. Now the Eq. (6) has made the Eq. (2) to the normal procedure of Hirota method for getting the exact soliton solutions. By making a series of suitable assumption for the expression of G and F, the exact one- and two-soliton solution can be obtained analytically. In order to obtain the bright one-soliton solution of Eq. (2) which correspond to the case $a < 0$, we proceed in the standard assumption

$$G = \chi G_1, \ F = 1 + \chi^2 F_1, \tag{7}$$

where χ is an arbitrary parameter which will be absorbed in expressing the soliton solution in the following sections. Substituting Eq. (7) into Eq. (6), then collecting the coefficients with same power in χ, we have

(1) for the coefficient of χ

$$\left(iD_t + \frac{1}{2}D_x^2 + xf(t)\right)G_1 \cdot 1 = 0, \tag{8}$$

(2) for the coefficient of χ^2

$$D_x^2 F_1 - \mu \overline{G}_1 G_1 = 0, \tag{9}$$

(3) for the coefficient of χ^3

$$\left(iD_t + \frac{1}{2}D_x^2 + xf(t)\right)G_1 \cdot F_1 = 0, \tag{10}$$

(4) for the coefficient of χ^4

$$D_x^2 F_1 \cdot F_1 = 0. \tag{11}$$

Using the definition (5) the above equations can be expressed in detail. For example, in order to satisfy Eq. (8) we can assume G_1 has the form

$$G_1 = e^{\eta_1}. \tag{12}$$

Substituting Eq. (12) into Eq. (8) we have

$$i\eta_{1t} + \frac{1}{2}\eta_{1xx} + \frac{1}{2}\eta_{1x}^2 + xf(t) = 0. \tag{13}$$

Because of the presence of the term $xf(t)$ there are some difficulties for getting general solutions of the above equation. For some conveniences in this paper we assume η_1 has the form

$$\eta_1 = P_1(t)x + \Omega_1(t), \tag{14}$$

with the time-dependent functions $P_1(t)$ and $\Omega_1(t)$ to be determined. With the restriction, i.e., Eqs. (9) and (11) we get

$$F_1 = \exp(\eta_1 + \overline{\eta}_1 + A_{11}), \tag{15}$$

where

$$A_{11} = \ln \frac{\mu}{\left(P_1 + \overline{P}_1\right)^2}.$$

Substituting the solutions (12) and (15) into Eq. (10) we obtain the equations of $P_1(t)$ and $\Omega_1(t)$ as

$$[iP_{1,t}(t) + f(t)]\,x + i\Omega_{1,t} + \frac{1}{2}P_1^2 = 0, \tag{16}$$

form which we can determine the expression of $P_1(t)$ and $\Omega_1(t)$. It is obvious that Eq. (16) implies the natural conditions

$$iP_{1,t}(t) + f(t) = 0,$$
$$i\Omega_{1,t}(t) + \frac{1}{2}P^2(t) = 0. \tag{17}$$

Solving the above two equations, we get the expression of P_1 and Ω_1 as

$$P_1 = i\int_0^t f(\tau)d\tau + \xi_{10},$$
$$\Omega_1 = \frac{i}{2}\int_0^t P_1^2(\tau)\,d\tau + \zeta_{10}, \tag{18}$$

where ξ_{10} and ζ_{10} are complex parameters in general. With the Eqs. (12), (15) and (3), after absorbing χ, the bright one-soliton solution of Eq. (2) can be derived as

$$\psi = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \bar{\eta}_1 + A_{11}}}, \tag{19}$$

where

$$\eta_1 = \left[i\int_0^t f(\tau)d\tau + \xi_{10}\right]x + \frac{i}{2}\int_0^t P_1^2(\tau)\,d\tau + \zeta_{10}.$$
$$A_{11} = \ln\frac{\mu}{\left(\xi_{10} + \bar{\xi}_{10}\right)^2}. \tag{20}$$

In the case of $f(t) = 0$, one-soliton solution (19) can reduce to the solutions of the normal nonlinear Schrődinger equation. When $f(t) = constant$ the solution (19) is the same results ([23]) reported earlier. As $f(t) = b_1 + l\cos(\omega t)$, the solution (36) is the same results ([24]). From Eq. (20) we can see the linear time-dependent potential can change the soliton velocity and frequency.

III.   TWO-SOLITON SOLUTION

In this section we will give the analytical expression of two soliton solution of Eq. (2). To this purpose we now assume that

$$G = \chi G_1 + \chi^3 G_2, \; F = 1 + \chi^2 F_1 + \chi^4 F_2. \tag{21}$$

By employing the same procedure before we obtain the following set of equations from Eq. (6), corresponding to the different powers of χ

(1) for the coefficient of χ

$$\left(iD_t + \frac{1}{2}D_x^2 + xf(t)\right) G_1 \cdot 1 = 0, \tag{22}$$

(2) for the coefficient of χ^2

$$D_x^2 F_1 \cdot 1 = \mu \overline{G}_1 G_1, \tag{23}$$

(3) for the coefficient of χ^3

$$\left(iD_t + \frac{1}{2}D_x^2\right)(G_1 \cdot F_1 + G_2 \cdot 1) + xf(t)(G_1 F_1 + G_2) = 0, \tag{24}$$

(4) for the coefficient of χ^4

$$D_x^2 F_1 \cdot F_1 + 2D_x^2 F_2 \cdot 1 = 2\mu(\overline{G}_1 G_2 + \overline{G}_2 G_1), \tag{25}$$

(5) for the coefficient of χ^5

$$\left(iD_t + \frac{1}{2}D_x^2\right)(G_1 \cdot F_2 + G_2 \cdot F_1) + xf(t)(G_1 F_2 + G_2 F_1) = 0, \tag{26}$$

(6) for the coefficient of χ^6

$$D_x^2 F_1 \cdot F_2 = \mu \overline{G}_2 G_2, \tag{27}$$

(7) for the coefficient of χ^7

$$\left(iD_t + \frac{1}{2}D_x^2 + xf(t)\right) G_2 \cdot F_2 = 0, \tag{28}$$

(8) for the coefficient of χ^8

$$D_x^2 F_2 \cdot F_2 = 0, \tag{29}$$

As discussed in the one-soliton solution, we can solve the equations from (22) to (29) in turn for getting the expression of G and F. In order to construct the two-soliton solution of (2) we assume G_1 has the form

$$G_1 = \exp \eta_1 + \exp \eta_2, \tag{30}$$

where

$$\eta_j = P_j(t)x + \Omega_j(t),$$

in which the time-dependent functions $P_j(t)$ and $\Omega_j(t)$, $j = 1, 2$, to be determined. Substituting Eq. (31) into the relation (22) we have

$$\left\{ [iP_{1,t} + f(t)]\, x + i\Omega_{1,t} + \frac{1}{2}P_1^2 \right\} \exp \eta_1 + \left\{ [iP_{2,t} + f(t)]\, x + i\Omega_{2,t} + \frac{1}{2}P_2^2 \right\} \exp \eta_2 = 0,$$

which implies that

$$iP_{j,t} + f(t) = 0,$$
$$i\Omega_{j,t} + \frac{1}{2}P_j^2 = 0.$$

where $j = 1, 2$. From the above equations one can find the solutions

$$P_j(t) = i \int_0^t f(\tau)d\tau + \xi_{j0},$$
$$\Omega_j(t) = \frac{i}{2} \int_0^t P_j^2(\tau)\, d\tau + \zeta_{j0},$$

where ξ_{j0} and ζ_{j0}, $j = 1, 2$, are complex parameters in general. Combining Eq. (30) with Eq. (23) we obtain the expression of F_1 as

$$F_1 = \exp(\eta_1 + \overline{\eta}_1 + A_{11}) + \exp(\eta_2 + \overline{\eta}_1 + A_{21})$$
$$+ \exp(\eta_1 + \overline{\eta}_2 + A_{12}) + \exp(\eta_2 + \overline{\eta}_2 + A_{22}), \tag{31}$$

where

$$A_{mn} = \ln \frac{\mu}{\left(P_m + \overline{P}_n\right)^2}, \quad m, n = 1, 2.$$

With the help of Eqs. (30) and (31) we can simplify Eq. (24) as

$$iG_{2t} + \frac{1}{2}G_{2xx} + xf(t)G_2$$
$$= \frac{\mu\,(P_2 - P_1)^2}{\left(P_1 + \overline{P}_1\right)\left(P_2 + \overline{P}_1\right)}e^{\eta_1 + \bar{\eta}_1 + \eta_2} + \frac{\mu\,(P_2 - P_1)^2}{\left(P_1 + \overline{P}_2\right)\left(P_2 + \overline{P}_2\right)}e^{\eta_1 + \bar{\eta}_2 + \eta_2}, \tag{32}$$

which shows that the expression of G_2 has the form

$$G_2 = \exp(\eta_1 + \overline{\eta}_1 + \eta_2 + \delta_1) + \exp(\eta_1 + \overline{\eta}_2 + \eta_2 + \delta_2), \tag{33}$$

where the parameter δ_j, $j = 1, 2$, is given by

$$\delta_j = \ln \left(\frac{B_j}{\gamma_j} \right), j = 1, 2, \tag{34}$$

$$\gamma_1 = \left(P_1 + \overline{P}_1\right)\left(P_2 + \overline{P}_1\right), \gamma_2 = \left(P_1 + \overline{P}_2\right)\left(P_2 + \overline{P}_2\right),$$

$$B_1 = \frac{\mu\left(P_2 - P_1\right)^2}{\left(P_1 + \overline{P}_1\right)\left(P_2 + \overline{P}_1\right)}, \quad B_2 = \frac{\mu\left(P_2 - P_1\right)^2}{\left(P_1 + \overline{P}_2\right)\left(P_2 + \overline{P}_2\right)}.$$

Now we have obtained the expression of G_1, G_2, and F_1 in Eq. (21). Substituting Eqs. (30), (31) and (33) into Eq. (25) we obtain

$$F_2 = e^{\eta_1 + \overline{\eta}_1 + \eta_2 + \overline{\eta}_2 + \kappa}, \tag{35}$$

where

$$\kappa = \ln \frac{\mu^2 |P_2 - P_1|^4}{\left(P_1 + \overline{P}_1\right)^2 \left(P_2 + \overline{P}_2\right)^2 |P_1 + \overline{P}_2|^4}.$$

With the help of Eqs. (30), (31), (33), and (35) one can find the Eqs. (26) and (29) are satisfied to the moment after a tedious calculation. So the two-soliton solutions has the form

$$\psi_2 = \frac{G}{F}, \tag{36}$$

where

$$G = e^{\eta_1} + e^{\eta_2} + e^{\eta_1 + \overline{\eta}_1 + \eta_2 + \delta_1} + e^{\eta_1 + \overline{\eta}_2 + \eta_2 + \delta_2},$$

$$F = 1 + e^{\eta_1 + \overline{\eta}_1 + A_{11}} + e^{\eta_2 + \overline{\eta}_1 + A_{21}} + e^{\eta_1 + \overline{\eta}_2 + A_{12}} + e^{\eta_2 + \overline{\eta}_2 + A_{22}} + e^{\eta_1 + \overline{\eta}_1 + \eta_2 + \overline{\eta}_2 + \kappa}.$$

$$P_j\left(t\right) = i \int_0^t f(\tau)d\tau + \xi_{j0},$$

$$\Omega_j(t) = \frac{i}{2} \int_0^t P_j^2\left(\tau\right)d\tau + \zeta_{j0},$$

$$\delta_j = \ln\left(\frac{B_j}{\gamma_j}\right),$$

where $j = 1, 2$, and

$$\gamma_1 = \left(\xi_{10} + \overline{\xi}_{10}\right)\left(\xi_{20} + \overline{\xi}_{10}\right),$$

$$\gamma_2 = \left(\xi_{10} + \overline{\xi}_{20}\right)\left(\xi_{20} + \overline{\xi}_{20}\right),$$

$$B_1 = \frac{\mu\left(\xi_{20} - \xi_{10}\right)^2}{\left(\xi_{10} + \overline{\xi}_{10}\right)\left(\xi_{20} + \overline{\xi}_{10}\right)},$$

$$B_2 = \frac{\mu\left(\xi_{20} - \xi_{10}\right)^2}{\left(\xi_{10} + \overline{\xi}_{20}\right)\left(\xi_{20} + \overline{\xi}_{20}\right)},$$

$$\kappa = \ln \frac{\mu^2 \left|\xi_{20} - \xi_{10}\right|^4}{\left(\xi_{10} + \overline{\xi}_{10}\right)^2 \left(\xi_{20} + \overline{\xi}_{20}\right)^2 \left|\xi_{10} + \overline{\xi}_{20}\right|^4},$$

$$A_{mn} = \ln \frac{\mu}{\left(\xi_{m0} + \overline{\xi}_{n0}\right)^2}, \ m, n = 1, 2.$$

When $f(t) = 0$, the solution (36) denotes two soliton interaction of the normal nonlinear Schrödinger equation. The new expression (36) implies that Hirota method has more advantage for getting new soliton solutions as well.

IV.   CONCLUSION

In this paper, we investigate the soliton solutions of the nonlinear Schrödinger equation with an arbitrary time-dependent linear potential which denotes the dynamics of quasi-one-dimensional Bose-Einstein condensation. A developed Hirota method is applied carefully to the nonlinear Schrödinger equation. In terms of this developed technique we decoupled the nonlinear Schrödinger equation into two equations. Moreover, with a reasonable assumption the exact one- and new two-soliton solutions are constructed effectively. Our soliton interaction will have useful application in the studies of Bose-Einstein condensation and optics communication in which the nonlinear Schrödinger equation are used widely.

V.   ACKNOWLEDGEMENT

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[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Science **269** (1995) 198;

K. B. Davis, M.-O. Mewes, M. R. Andrews, N. J. van Druten, D. S. Durfee, D. M. Kurn, and W. Ketterle, Phys. Rev. Lett. **75** (1995) 3969.

[2] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Phys. Rev. Lett. **75** (1995) 1687; **78** (1997) 985;

D. G. Fried, T. C. Killian, L. Willmann, D. Landhuis, S. C. Moss, D. Kleppner, and T. J. Greytak, Phys. Rev. Lett. **81** (1998) 3811.

[3] L. Deng, E. W. Hagley, J. W. En, M. Trippenbach, Y. Band, P. S. Julienne, J. E. Simsarian, K. Helmerson, S. L. Rolston, and W. D. Phillips, Nature (London) **398** (1999) 218;

Y. Wu and X. Yang, J. Opt. Soc. Am. B 23 (2006) 1888; Optics Letters 30, No. 3 (2005) 311; Phys. Rev. A. 68 (2003) 013608.

[4] A. L. Fetter and A. A. Svidzinsky, J. Phys.: Condens. Matter **13** (2001) R135.

[5] R. J. Donnely, Quantized Vortices in Helium II (Cambridge University Press, Cambridge, 1991).

[6] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83** (1999) 2498.

[7] Y. S. Kivshar and B. Luther-Davies, Phys. Rep. **298** (1998) 81.

[8] P. A. Ruprecht, M. J. Holland, K. Burnett, and M. Edwards, Phys. Rev. A **51** (1995) 4704.

[9] W. Zhang, D. F. Walls, and B. C. Sanders, Phys. Rev. Lett. **72** (1994) 60;

W. P. Reinhardt and C. W. Clark, J. Phys. B **30** (1997) L785;

A. D. Jackson, G. M. Kavoulakis, and C. J. Pethick, Phys. Rev. A **58** (1998) 2417.

[10] A. E. Muryshev, H. B. van Linden van den Heuvell, and G.V. Shlyapnikov, Phys. Rev. A **60** (1999) R2665;

Lincoln D Carr, Mary Ann Leung, and William P Reinhardt, J. Phys. B **33** (2000) 3983.

[11] J. Denschlag, et al., Science **287** (2000) 97.

[12] J. C. Bronski, L. D. Carr, B. Deconinck, and J. N. Kutz, Phys. Rev. Lett. **86** (2001) 1402.

[13] L. D. Carr, C. W. Clark, and W. P. Reinhardt, Phys. Rev. A **62** (2000) 063610; **62** (2000) 063611;

D. L. Feder et al., Phys. Rev. A **62** (2000) 053606.

[14] R. Dum, J. I. Cirac, M. Lewenstein, and P. Zoller, Phys. Rev. Lett. **80** (1998) 2972.

[15] S. Burger, K. Bongs, S. Dettmer, W. Ertmer, and K. Sengstock, Phys. Rev. Lett. **83** (1999) 5198.

[16] Kevin E. Strecker, Guthrie B. Partridge, Andrew G. Truscott, and Randall G. Hulet, Nature (London) 417 (2002) 150.

[17] L. Khaykovich, F. Schreck, G. Ferrari, T. Bourdel, J. Cubizolles, L. D. Carr, Y. Castin, and C. Salomon, Science **296** (2002) 1290.

[18] U. Al Khawaja, H. T. C. Stoof, R. G. Hulet, K. E. Strecker, and G. B. Partridge, Phys. Rev. Lett. **89** (2002) 200404.

[19] Shun-Jin Wang, Cheng-Long Jia, Dun Zhao, Hong-Gang Luo, and Jun-Hong An, Phys. Rev. A **68** (2003) 015601.

[20] Zai-Dong Li, P. B. He, Lu Li, J.-Q. Liang, and W. M. Liu, Phys. Rev. A 71 (2005) 053611;
Lu Li, Zaidong Li, Boris A. Malomed, Dumitru Mihalache, and W. M. Liu, Phys. Rev. A **72** (2005) 033611;
Q. Y. Li, Z. W. Xie, L. Li, Z. D. Li, and J. Q. Liang, Annals of Physics 312 (2004) 128.

[21] A. Görlitz, J. M. Vogels, A. E. Leanhardt, C. Raman, T. L. Gustavson, and J. R. Abo-Shaeer, Phys. Rev. Lett. **87** (2001) 130402;
S. Dettmer et al., ibid. **87** (2001) 160406.

[22] D. S. Petrov, G.V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85** (2000) 3745.

[23] H. H. Chen and C. S. Liu, Phys. Rev. Lett. **37** (1976) 693;
V. V. Konotop, O. A. Chubykalo, and L, Vzquez, Phys. Rev. E **48** (1993) 563.

[24] Qin Yang, and Jie-fang Zhang, Optics Communications **258** (2006) 35.